

# $AdS$ null deformations with inhomogeneities

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## Abstract

We study  $AdS \times X$  null deformations arising as near horizon limits of D3-brane analogs of inhomogeneous plane waves. Restricting to normalizable deformations for the  $AdS_5$  case, these generically correspond in the dual field theory to SYM states with lightcone momentum density  $T_{++}$  varying spatially, the homogeneous case studied in arXiv:1202.5935 [hep-th] corresponding to uniform  $T_{++}$ . All of these preserve some supersymmetry. Generically these inhomogeneous solutions exhibit analogs of horizons in the interior where a timelike Killing vector becomes null. From the point of view of  $x^+$ -dimensional reduction, the circle pinches off on these horizon loci in the interior. We discuss similar inhomogeneous solutions with asymptotically Lifshitz boundary conditions, as well as aspects of Lifshitz singularities in string constructions involving  $AdS$  null deformations. We also briefly discuss holographic entanglement entropy for some of these.

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## 1 Introduction

Various fascinating explorations of strongly coupled quantum field theories have been carried out using gauge/gravity duality [1], including, more recently, non-relativistic and condensed matter systems [2] with symmetries typically smaller than anti de Sitter space. Several interesting features of finite density systems can in fact be simulated in fairly simple effective gravity models with cosmological constant and vector/scalar matter sources. It is important to understand such models in string theory: for one thing, it is expected that the parameter space of string constructions is more constrained, and it might be possible to track the stringy origins of the effective parameters. Furthermore, a string/brane construction might suggest natural field theory duals to the gravity descriptions.

A remarkably simple family of such string realizations of some nonrelativistic systems involves null deformations of  $AdS \times X$  spacetimes that arise in familiar brane constructions, of the form

$$ds^2 = \frac{1}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + g_{++}(dx^+)^2 + d\Omega_S^2, \quad (1)$$

where the metric component  $g_{++}$  might be sourced by various fields. For instance, spacetimes with  $z = 2$  Lifshitz scaling symmetry [3, 4] can be realized in string constructions via dimensional reduction along the  $x^+$ -direction of metrics (1) with non-normalizable deformations  $g_{++} \sim \frac{1}{r^2}r^2$  [5, 6] (see also *e.g.* [7]). Likewise, metrics conformal to Lifshitz spacetimes arise in holographic systems with nontrivial hyperscaling violation exponents encoded in the conformal factor: these have been discussed in *e.g.* [8, 9] motivating and clarifying connections with condensed matter systems (in particular pertaining to holographic entanglement entropy [10] and Fermi surfaces): see also [11] for various aspects of holography in this context. Some of these hyperscaling violating metrics can be realized by  $x^+$ -dimensional reduction of metrics (1) using normalizable deformations  $g_{++} \sim \frac{1}{r^2}r^4$  [12] (see also *e.g.* [11, 13, 14, 15, 16, 17] for other string realizations of such models, and *e.g.* [18, 19, 20, 21, 22, 23, 24, 25] for effective gravity models with vectors and scalars where these and related metrics arise). In particular, the  $AdS_5$  normalizable null deformation after dimensional reduction interestingly

gives a spacetime with  $\theta = 1, d = 2$ , which exhibits logarithmic violation of the area law of entanglement entropy in the holographic context.

In this paper, we will explore the space of such  $AdS \times X$  null deformations in greater generality, allowing for possible inhomogeneities, *i.e.* with  $g_{++}$  having spatial ( $x_i$ ) dependence. These are near horizon limits of D3-brane analogs of plane-waves with possible inhomogeneities (see *e.g.* [26] for a recent review discussing plane-wave backgrounds [27] in the context of cosmological singularities). Since  $AdS_5 \times S^5$  is  $\alpha'$ -exact [28] as are plane wave spacetimes, these  $AdS$  null deformations are also likely  $\alpha'$ -exact string backgrounds. We will restrict in particular to static  $AdS$  deformations that are normalizable near the boundary, *i.e.* normalizable backgrounds, so they can be interpreted as states in the dual super Yang-Mills theory with nontrivial lightcone momentum density  $T_{++}$  that might vary spatially (regarding  $x^+$  as a noncompact direction, sec. 2). In general, the structure of these normalizable background solutions involves modes that grow in the interior: this implies that for large families of solutions with inhomogeneities,  $g_{++}$  vanishes somewhere in the interior, even if it is positive definite near the boundary. These  $g_{++} = 0$  loci are akin to horizons, in the sense that a timelike Killing vector  $\partial_-$  becomes null.

Part of the motivation here is to understand the “vicinity” of the homogenous  $AdS$  plane wave studied in [12]. In other words, we would like to explore “nearby” solutions, at least within the class describable as  $AdS$  null deformations. From the dual field theory point of view, the homogenous case has uniform lightcone momentum density  $T_{++} \sim Q$ , so that  $T_{++} \sim Q + \epsilon f(x, y)$  with small  $\epsilon$  (and localized near some  $x_{i0}$ ) would constitute a “small” inhomogenous perturbation. Then we find that starting with a homogenous background  $g_{++}$ , turning on a “small” inhomogenous perturbation near the boundary (*i.e.*  $T_{++}$  as above) corresponds to a bulk spacetime which departs substantially from the homogenous  $AdS$  plane wave, due to the emergence of horizon loci.

From the perspective of  $x^+$ -dimensional reduction (sec. 3) of such backgrounds, it appears that the circle pinches off at the loci where  $g_{++} = 0$ , and new states emerge corresponding to string winding modes that become light in the vicinity of these loci. We also explore asymptotically Lifshitz backgrounds with inhomogeneities, and discuss Lifshitz singularities (arising from diverging tidal forces in the interior) from the point of view of string constructions involving  $AdS$  null deformations.

Finally, we discuss holographic entanglement entropy briefly for these null deformed backgrounds (sec. 4) from the point of view of the higher dimensional description (*i.e.* with  $x^+$ -noncompact).

Our discussion is primarily for the  $AdS_5$  case arising from D3-branes: we also briefly analyse other  $AdS_D$  null deformations of this sort (sec. 2), and expect similar features.

## 2 $AdS_5$ null deformations with inhomogeneities

We are considering spacetimes of the form (1), *i.e.*

$$ds^2 = \frac{1}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + g_{++}(dx^+)^2 + d\Omega_S^2, \quad (2)$$

as solutions to IIB string theory (or supergravity), obtained by null deformations of the familiar near horizon geometry of a D3-brane stack. Our discussion will be mostly for  $AdS_5 \times S^5$  dual to 4-dim  $\mathcal{N}=4$  SYM theory, but the arguments can also be generalized to other super Yang-Mills theories dual to  $AdS_5 \times X^5$  spacetimes, with the 5-space  $X^5$  being a Sasaki-Einstein base or equivalently the 6-dim space transverse to the D3-branes being Ricci-flat. More generally, the 10-dim spacetime

$$ds^2 = Z^{-1/2}[-2dx^+dx^- + dx_i^2 + N(x_i, x^m)(dx^+)^2] + Z^{1/2}dx^mdx^m, \quad (3)$$

$Z(x^m)$  being harmonic in the transverse space, with the corresponding 5-form flux describing the stack of D3-branes with null deformation gives in the near horizon limit the metrics (2) above for a single D3-brane stack. Without the  $Z$  factors, this spacetime is a solution if  $\partial_M \partial^M N = 0$ , where  $M = i, m$ : these are essentially plane waves with inhomogeneities, and including the  $Z$  factors gives D3-brane analogs thereof. Recalling that both  $AdS_5 \times S^5$  and plane wave spacetimes are  $\alpha'$ -exact string backgrounds, it is likely that the backgrounds (2), and (3) in the near horizon limit, are also  $\alpha'$ -exact. To elaborate on this, we note that the curvature invariants  $R$ ,  $R_{MN}R^{MN}$ ,  $R_{MNPQ}R^{MNPQ}$ , for these null-deformed spacetimes are finite and identical to those of  $AdS_5 \times S^5$ . Any higher derivative correction to the action is expected to stem from covariant contractions involving the additional curvature components: however since the metric itself (and also *e.g.* the Ricci tensor etc) has no nonzero component with multiple upper  $+$ -indices, these additional contractions vanish, thus giving no further corrections to  $AdS_5 \times S^5$  which itself is  $\alpha'$ -exact [28]. While this is not a proof that all higher derivative corrections to (2) vanish, it is suggestive, making these  $AdS$  plane waves potentially more interesting.

We will mainly focus here on null deformations with  $g_{++}(x_i, r)$ , which can all be thought of as simply solutions to 5-dim gravity with negative cosmological constant, satisfying  $R_{MN} = -4g_{MN}$  ( $M, N = \mu, r$ ), the cosmological constant arising from the 5-form flux, the  $X^5$  part effectively untouched. Thus they are all solutions to a 5-dim effective action  $S_5 \sim \int d^5x \sqrt{-g} (R+12)$ . [More generally, we can consider  $g_{++}(r, x_i, x^+, \Omega_l)$ , sourced by other matter fields.] The 5-dim part of the spacetimes (2) are then solutions to  $R_{MN} = -4g_{MN}$  if  $g_{++}$  satisfies

$$r^2 \partial_r^2 g_{++} + r \partial_r g_{++} - 4g_{++} + r^2 \partial_i^2 g_{++} = 0. \quad (4)$$

$\partial_-$ ,  $\partial_+$  are Killing vectors if  $g_{++} = g_{++}(r, x_i)$ . With  $g_{++} > 0$ , it is natural to take  $x^-$  as the time direction ( $g^{--} = -r^4 g_{++} < 0$  implies that  $x^-$ -constant surfaces are spacelike), with  $x^+$  then a spatial direction. The deformation being lightlike is special: for instance, it is noteworthy that the equation above is linear, although this is in the full nonlinear gravity theory (not just in a linearized approximation). It is also noteworthy that turning on the  $g_{++}$  mode (keeping it static in  $x^-$ -time) does not source any other metric component: this is a consistent closed subsystem in itself. All these solutions preserve some supersymmetry, as will be outlined in the next subsection: in particular, the inhomogeneous solutions preserve the same amount of supersymmetry as the homogeneous one, so in some sense there is a moduli space of solutions here (although in a reduced effective action,  $g_{++}$  does not enter, being lightlike).

We want to focus on normalizable solutions for  $g_{++}$ , *i.e.*  $g_{++} \rightarrow \frac{1}{r^2} r^4 f_4(x_i)$  near the boundary  $r = 0$ . Using the usual *AdS/CFT* dictionary, these will then have the interpretation of states in the  $\mathcal{N}=4$  SYM theory. To elaborate on this, consider an asymptotically *AdS*<sub>5</sub> solution with metric of the form

$$ds^2 = \frac{dr^2}{r^2} + h_{\mu\nu} dx^\mu dx^\nu = \frac{dr^2}{r^2} + \frac{1}{r^2} (g_{\mu\nu}^{(0)} + r^2 g_{\mu\nu}^{(2)} + r^4 g_{\mu\nu}^{(4)} + \dots) dx^\mu dx^\nu, \quad (5)$$

written in the Fefferman-Graham expansion about the boundary  $r = 0$ . Then holographic renormalization methods [29, 30, 31, 32, 33] give relations between the metric coefficients  $g_{\mu\nu}^{(k)}$  and the holographic stress tensor calculated as  $T_{\mu\nu} \sim \lim_{r \rightarrow 0} \frac{1}{r^2} \frac{1}{G_5} (K_{\mu\nu} - K h_{\mu\nu} - 3 h_{\mu\nu})$ , where  $K_{\mu\nu}$  is the extrinsic curvature<sup>1</sup>. For (2), the only departures from the *AdS*<sub>5</sub> expressions are in  $\{++\}$ -components and we have

$$K_{++} = r^2 f_4(x_i) \quad \Rightarrow \quad T_{++} = \frac{f_4(x_i)}{4\pi G_5}. \quad (6)$$

This stress tensor, roughly encoding a “chiral” wave, is automatically traceless and conserved. Thus these null deformed spacetimes all correspond to waves on the boundary with nonzero constant energy momentum component  $T_{++}$ , varying inhomogeneously in the  $x_i$ -plane. In other words, these correspond to lightcone states in  $\mathcal{N}=4$  SYM theory, obtained by turning on finite lightcone momentum density, possibly with inhomogeneities. The homogeneous *AdS* plane wave solution studied in [12] corresponds to uniform lightcone momentum density. The total lightcone momentum is  $P_+ \sim \int d^2 x_i dx^+ T_{++}$ . No other observable has an expectation value in the dual field theory in these states. The fact that the  $g_{++}$ -mode comprises a closed subsystem in itself appears to be a reflection of the fact that lightcone momentum density (constant in lightcone time) can be consistently turned on in lightcone SYM without sourcing

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<sup>1</sup> We have  $K_{\mu\nu} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$ , with  $n_\mu$  the outward pointing unit normal to the surface  $r = \text{const.}$  For the boundary being  $r = 0$ , we have  $n = -\frac{dr}{r}$ : this gives  $K_{\mu\nu} = \frac{r}{2} h_{\mu\nu,r}$ .

other operators. It is worth noting that other  $x^+$ -dependent deformations of  $AdS/CFT$  may well exist, with nonzero energy-momentum component  $T_{++}$ , suggesting that the present discussion is not a complete classification of lightcone states in  $\mathcal{N}=4$  SYM: however these will generically have other energy-momentum components nonzero too, thus differing from the class here.

One might expect the local lightcone momentum density in the field theory in a region  $\Delta x_i$  to always be positive, since the local dispersion relation  $p_- = \frac{p_i^2}{2p_+}$ , where  $p_+ \sim \int_{\Delta x_i} T_{++}$ , would suggest negative energy states otherwise. Imposing this, we have

$$T_{++} \geq 0 \quad (7)$$

everywhere in space: this also effectively follows from a null energy condition on the boundary. This is a nontrivial physical condition on the solutions, as we see below. Another physical criterion is that the energy-momentum density is bounded: this is also not generically true, and imposing this restricts the family of solutions.

Towards understanding solutions, we note that the  $g_{++}$ -equation is linear: this means that various “basis” solutions can be superposed to give composite solutions. To find the basis solutions, consider a separable ansatz  $g_{++} = h_{++}(r)f(x, y)$ , which gives

$$r^2 h''_{++} + r h'_{++} - (4 + k^2 r^2) h_{++} = 0, \quad \partial_i^2 f + k^2 f = 0. \quad (8)$$

for some constant parameter  $k$ . For  $k = 0$  and  $f = \text{const}$  (*i.e.* no  $x_i$ -dependence), we have  $g_{++} = h_{++} = Qr^2$  as the normalizable solution: this is the homogenous  $AdS$  plane wave discussed in [12], with  $T_{++} \geq 0$  implying  $Q \geq 0$ . The non-normalizable solution  $g_{++} \sim \frac{1}{r^2}$  is in fact simply  $AdS$  after a coordinate transformation. Inhomogenous solutions also exist for  $k = 0$ , as we will see below.

For nonzero  $k$ , the radial equation is a Bessel equation in general. A subset of basis modes is then obtained with  $k^2 = k_1^2 + k_2^2 > 0$ : this gives  $f(x, y) = \sin k_1 x \sin k_2 y$ , and  $h_{++}(r) = K_2(kr), I_2(kr)$ . Restricting to normalizable solutions (noting the asymptotics  $I_2(kr) \sim k^2 r^2$  as  $r \rightarrow 0$ ) picks out modes of the form  $g_{++} = I_2(kr) \sin k_1 x \sin k_2 y$  (or  $g_{++} = I_2(kr) e^{ik_1 x + ik_2 y}$ ). Thus for given  $(k_1, k_2)$ , the  $x_i$ -plane is lattice-like in a sense, with a unit cell of size  $\sim \frac{1}{k_1} \times \frac{1}{k_2}$ . The general solution of this sort, and its asymptotic form near the boundary  $r = 0$ , is

$$g_{++} = \int d^2 k_i \tilde{f}(k_1, k_2) I_2(kr) e^{ik_1 x + ik_2 y} \xrightarrow{r \rightarrow 0} r^2 \int d^2 k_i \tilde{f}(k_1, k_2) k^2 e^{ik_1 x + ik_2 y}, \quad (9)$$

the  $k^2 \tilde{f}(k_1, k_2)$  being dimensionful (complex) Fourier coefficients of the function  $f_4(x_i)$  (using (6)) over the plane wave components  $e^{ik_i x_i}$ : the homogenous solution is given by  $\tilde{f}(k_1, k_2) =$

$\frac{Q}{k^2}\delta^2(k_i)$ . In general, the coefficients  $k^2\tilde{f}(k_1, k_2)$  have dimensions of (boundary) energy density. A generic function  $f_4(x_i)$  can be constructed from this basis, giving the most general lightcone momentum density  $T_{++} \sim f_4(x_i)$  (6). In other words, a general spatially varying  $T_{++}$  maps to a (static) bulk dual metric with  $g_{++}$  as above.

To put the current discussion in perspective with that on normalizable modes in *e.g.* [34], consider a massless scalar field  $\varphi$  in  $AdS_5$  in lightcone coordinates: then  $\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\varphi) = 0$  for modes  $\varphi = e^{ik_+x^+ + ik_-x^- + ik_ix_i}R(r)$  simplifies to

$$R'' - \frac{3}{r}R' - k^2R = 0, \quad k^2 \equiv k_\mu k^\mu = -2k_+k_- + k_i^2. \quad (10)$$

Redefining  $R = r^2y(r)$  casts this radial equation as  $r^2y'' + ry' - (4 + k^2r^2)y = 0$ , *i.e.* in the same form as that in (8). Now with  $k^2 = 0$ , we have  $y = r^2$  as the normalizable solution. With  $k^2 < 0$ , this gives  $y = J_2(kr)$  as normalizable modes: however  $k^2 = -2k_+k_- + k_i^2 < 0$ , which requires  $k_\pm \neq 0$ , so these are fluctuating modes. If we look for static,  $x^-$ -independent modes, then we have  $k^2 = k_i^2 \geq 0$ , and the radial equation gives  $y = I_2(kr)$  as the normalizable solution for nonzero  $k^2$ , *i.e.*  $\varphi = r^2I_2(kr)e^{ik_ix_i}$  (or  $\varphi = r^4$  for  $k^2 = 0$ ). These are thus normalizable backgrounds (rather than fluctuating modes), which have the interpretation of time-independent states in the dual gauge theory. Generic metric perturbations (at linearized order) are governed by a similar equation, so their behaviour for normalizable background solutions is similar. The  $g_{++}$ -mode is special, in that it is a closed subsystem in itself, with its equations at linearized order (4), (8), in fact being exact in the full nonlinear theory.

These normalizable background solutions to (8) grow exponentially in the interior, and one might be concerned if these inhomogenous solutions are physically allowed. In this light, we recall that the homogenous  $AdS$  plane wave with  $g_{++} = Qr^2$  also grows in the interior, although only as a power law. In general, from above, we see that a normalizable static background will grow in the interior. Also from the dual field theory point of view, we are describing configurations with nontrivial lightcone momentum density  $T_{++}$  distinct from the vacuum, the homogenous  $AdS$  plane wave corresponding to uniform  $T_{++}$ . The general spatially varying boundary  $T_{++}$  which is bounded, conserved and satisfying sensible energy conditions and thus an allowed configuration in field theory, maps quite generally as (9) above to these static normalizable backgrounds using these basis modes of (8), which are the only static normalizable solutions with  $k^2 > 0$  corresponding to oscillatory  $e^{ik_ix_i}$ -type boundary spatial behaviour. Small (linearized) fluctuations about  $AdS_5$  with only  $g_{++}$  nonzero (*i.e.* only  $T_{++}$  nonzero) of the form  $e^{ik_+x^+ + ik_-x^- + ik_ix_i}h_{++}(r)$  including lightcone time  $x^-$  dependence are also governed by (8) at linear order, the other equations forcing  $k_- = 0$ . This could mean that small  $g_{++}$ -fluctuations also source other modes at linear order: however it also implies that static  $g_{++}$ -backgrounds form a closed subsystem. For  $k^2 = k_i^2 < 0$  in (8),

$f(x_i)$  contains hyperbolic sinh/cosh-functions, the radial equation giving Bessel functions  $J_2(kr), Y_2(kr)$ . Then basis modes  $J_2(kr) \cosh k_1 x \cosh k_2 y$  are normalizable at the boundary. Also for  $k^2 = 0$ , we have modes  $r^2(x^2 - y^2)$  or  $r^2 \sin \chi x \sinh \chi y$  (for any  $\chi$ ) as inhomogenous normalizable backgrounds. However, in these cases, the boundary lightcone momentum density  $T_{++}$  for these basis modes is not bounded, growing indefinitely in certain regions for large  $x, y$ , which makes their status less clear.

From the asymptotics  $I_2(kr) \sim \frac{e^{kr}}{\sqrt{r}}$  ( $r \rightarrow \infty$ ), we have seen that these basis solutions always grow large in the interior  $r \rightarrow \infty$ : thus  $g_{++}$  is not always positive, since the sines oscillate and  $I_2$  grows for large  $r$ . Thus there exist loci where  $g_{++} = 0$ , quite generically: at these,  $g^{--} = -r^4 g_{++} = 0$ , so that constant- $x^-$  hypersurfaces which are spacelike for  $g_{++} > 0$  become null. Alternatively, the vector  $\partial_-$  which is timelike for  $g_{++} > 0$  becomes null at the loci  $g_{++} = 0$ . These loci are thus akin to *horizons*, where both  $x^\pm$  are lightlike directions. The vicinity of a horizon does not have anything unusual happening, the spacetime becoming simply  $AdS_5$  in lightcone coordinates: since curvature invariants are finite for the entire spacetime, they are in particular finite in the vicinity of the horizon too (it would be interesting to study geodesics and tidal forces in detail here). Generic particle trajectories will cross the horizon: on crossing a horizon,  $x^+$  becomes the natural time direction. Consider a bulk particle trajectory with action  $S = \frac{1}{2}m \int d\tau g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2}m \int d\tau (2g_{+-} \dot{x}^+ \dot{x}^- + g_{++} (\dot{x}^+)^2 + g_{IJ} \dot{x}^I \dot{x}^J)$ . If we fix lightcone gauge  $x^+ = \tau$  as often defined, we have the conjugate momenta  $p_I = m g_{IJ} \dot{x}^J$ ,  $p_- = \frac{\partial L}{\partial \dot{x}^-} = m g_{+-}$  ( $p_- < 0$  if  $g_{+-} < 0$ ), and the Hamiltonian

$$H = p_- \dot{x}^- + p_I \dot{x}^I - L = \frac{1}{2m} g^{IJ} p_I p_J - \frac{1}{2} m g_{++} = \frac{1}{2|p_-|} p_I^2 - \frac{|p_-|}{2} r^2 g_{++} ,$$

using  $g_{II} = -g_{+-} = \frac{1}{r^2}$ . We see that  $g_{++} < 0$  gives a positive potential while  $g_{++} > 0$  makes the potential term negative: the latter is in some sense an artifact of choosing lightcone gauge here fixing the spatial direction  $x^+$  as time. Approaching the horizon locus, we obtain a free particle Hamiltonian, with the dispersion relation  $H \equiv p_+ \sim \frac{1}{2|p_-|} p_I^2$ . Alternatively, fixing  $\tau = x^-$ , we have  $S = \frac{1}{2}m \int d\tau (2g_{+-} \dot{x}^+ \dot{x}^- + g_{++} (\dot{x}^+)^2 + g_{IJ} \dot{x}^I \dot{x}^J)$ , and the conjugate momenta  $p_I = m g_{IJ} \dot{x}^J$ ,  $p_+ = \frac{\partial L}{\partial \dot{x}^+} = m g_{+-} + m g_{++} \dot{x}^+$ . This gives the Hamiltonian  $H = p_+ \dot{x}^+ + p_I \dot{x}^I - L = \frac{1}{2m} g^{IJ} p_I p_J + \frac{1}{2m g_{++}} (p_+ - m g_{+-})^2$ . For  $g_{++} \rightarrow 0$  however, the last term arising from  $g_{++} (\dot{x}^+)^2$  disappears since now  $p_+ \rightarrow m g_{+-}$ , and we again recover the nonrelativistic dispersion relation  $H \equiv p_- = \frac{1}{2p_+} p_I^2$  near the  $g_{++} = 0$  horizon, as for  $AdS_5$  in lightcone coordinates.

For any such  $AdS$  plane wave, there are typically multiple such (disconnected) horizons, since the loci  $g_{++} = 0$  have many solutions. To illustrate this, consider for simplicity the solution

$$g_{++} = Qr^2 + \tilde{f}_k I_2(kr) \sin kx , \quad (11)$$



which has  $y$ -translations, but is  $x$ -striped. This is a superposition of a basis solution above and the homogenous  $AdS$  plane wave discussed in [12]. Near the boundary  $r \rightarrow 0$ , we have  $g_{++} \rightarrow r^2(Q + k^2 \tilde{f}_k \sin kx)$ , so that the holographic lightcone momentum density  $T_{++} \sim Q + k^2 \tilde{f}_k \sin kx$ , is positive for  $Q > k^2 \tilde{f}_k$ . However since  $I_2(kr)$  grows, a horizon  $g_{++} = 0$  develops in the interior: *e.g.* within the unit cell  $-\frac{\pi}{k} \leq x \leq \frac{\pi}{k}$ , the subregion  $-\frac{\pi}{k} \leq x \leq 0$  gives rise to a horizon locus  $g_{++} = 0$ . In the solution above, this is

$$\sin kx = -\frac{Qr^2}{\tilde{f}_k I_2(kr)} . \quad (12)$$

So we see that  $x \rightarrow 0$  as  $r \rightarrow \infty$  and  $x \rightarrow -\frac{\pi}{2k}$  as  $r$  approaches the value where the r.h.s. becomes unity. In the  $(r, x)$ -plane, these are roughly half-ellipse-shaped curves, one in the appropriate subregion in each unit cell. In the subregion  $0 \leq x \leq \frac{\pi}{k}$ , the sines are positive and  $g_{++} > 0$  so there is no horizon. For spacetimes with  $g_{++}$  having both  $x, y$ -dependence, the horizon loci are given by the surface  $g_{++}(r, x_i) = 0$ , or  $r_0 = r(x_i)$  as the implicit solutions.

Consider a spacetime of the form (2), with a leading homogenous plane wave piece superposed with a “small” inhomogeneity in the  $x$ -direction, thinking of the inhomogeneity as a static perturbation ( $y$ -translation symmetry exists). We would like to define this by requiring that the energy-momentum or lightcone momentum density  $T_{++}$  is only a small departure from constant density  $Q$ , *e.g.*  $T_{++} \sim Q + \epsilon f(x)$ . When the parameter  $\epsilon = 0$ , this is just the homogenous  $AdS$  plane wave. Now since  $T_{++}$  is essentially the asymptotic form of  $g_{++}$ , we effectively see that  $g_{++}$  can be found using the Fourier coefficients of  $f(x)$ , using (9). Consider thus Fourier modes of the form  $Ae^{-k^2\sigma^2}$ : these could reflect a small (Gaussian) lump localized around  $x = 0$  with width  $\sigma$ , and thus a small perturbation for small amplitude  $A$ . We then see that

$$g_{++} \sim Qr^2 + A \int_{-\infty}^{\infty} dk e^{-k^2\sigma^2} I_2(kr) \sin kx \xrightarrow{r \rightarrow \infty} Qr^2 + A \int_{-\infty}^{\infty} dk e^{-k^2\sigma^2} \frac{e^{kr}}{\sqrt{r}} \sin kx . \quad (13)$$

For  $x \rightarrow 0^+$ , each individual component in the second term is positive and  $g_{++} > 0$ . However evaluating this at  $x \rightarrow 0^-$  and approximating, we see that the sines are negative and the second term grows large and negative: thus we expect that  $g_{++} = 0$  somewhere. We see that  $x \rightarrow 0^\pm$  have substantially different behaviour for a small inhomogenous perturbation about  $x = 0$  near the boundary, with in fact large effects in the interior. This suggests that in fact the homogenous  $AdS$  plane wave is special: apparently “nearby”  $AdS$  plane waves with small inhomogenous modifications are in fact large departures with qualitative differences in the interior, such as the emergence of  $g_{++} = 0$  horizon loci.

It is also interesting to imagine a general near-boundary Fefferman-Graham expansion

$$g_{++} = \frac{1}{r^2}(r^4 f_4(x_i) + r^6 f_6(x_i) + \dots) \quad (14)$$

with the equation of motion (4) giving relations between the coefficients  $f_n(x_i)$ ,

$$12f_6 + \partial_i^2 f_4 = 0, \quad \dots \quad n(n-4)f_n + \partial_i^2 f_{n-2} = 0. \quad (15)$$

Then (with  $y$ -translations retained) there exist truncated solutions of the form  $g_{++} = r^2 f_4(x) + r^4 f_6(x)$ , where  $f_6'' = 0$ ,  $f_4'' + 12f_6 = 0$ : an example is  $g_{++} = Q(6r^2 x^2 - r^4)$ . This has  $T_{++} \sim Qx^2$ , and a  $g_{++} = 0$  horizon locus  $x^2 = \frac{1}{6}r^2$ , which intersects the boundary at  $r = 0$ , and so appears somewhat different from *e.g.* (12) which has support only in the interior.

Our discussion so far has been primarily for  $AdS_5$  null deformations.  $AdS_D$  null deformations of the form (2) are solutions if

$$r^2 \partial_r^2 g_{++} + (6-D)r \partial_r g_{++} - (2D-6)g_{++} + r^2 \partial_i^2 g_{++} = 0. \quad (16)$$

Modes with spatial dependence  $e^{ik_i x_i}$  give a radial equation  $r^2 h_{++}'' + (6-D)r h_{++}' - (2D-6 + r^2 k^2)h_{++} = 0$ . The normalizable background solutions are then  $r^{(D-5)/2} I_{\frac{D-1}{2}}(kr) e^{ik_i x_i}$ . Near the boundary, these asymptote to  $r^{D-3}$ , while for large  $r$ , we obtain exponential growth  $e^{kr}$ . For  $D = 3$ , there are no spatial directions  $x_i$ , and the equation above reduces to  $r^2 g_{++}'' + 3r g_{++}' = 0$ , which gives  $g_{++} = \text{const}$  as the normalizable solution, which is automatically homogenous (aspects of null or chiral deformations of conformal field theories appear in *e.g.* [35]). In general, these null deformations amount to turning on nontrivial expectation values for lightcone momentum density  $T_{++}$  in the dual conformal field theory: for M2- and M5-branes, these  $AdS_4 \times X^7$  and  $AdS_7 \times X^4$  plane waves would seem to correspond to states with lightcone momentum density  $T_{++}$  in the Chern-Simons (ABJM-like) and the (2,0) theories respectively. It would be interesting to explore these further.

We have been discussing normalizable backgrounds with broken translation invariance: in this context, it is worth noting [25] which discusses homogenous but anisotropic backgrounds which arise from extremal branes with a Bianchi classification.

## 2.1 Supersymmetry

We want to study supersymmetry properties of the 10-dim spacetime (3). Defining vielbeins

$$e^+ = Z^{-1/4} dx^+, \quad e^- = Z^{-1/4} (dx^- - \frac{N}{2} dx^+), \quad e^i = Z^{-1/4} dx^i, \quad e^m = Z^{1/4} dx^m, \quad (17)$$

which are natural for the lightcone coordinates here. (For  $N(x^i, x^m) = 0$ , this is simply the D3-brane stack solution in lightcone coordinates.) Using  $de^a = -\omega^a_b \wedge e^b$ , this then gives the

spin connection

$$\begin{aligned}
\omega_{-m} &= \frac{1}{4} Z^{-1/4} \partial_m \log Z e^+ , \quad \omega_{im} = -\frac{1}{4} Z^{-1/4} \partial_m \log Z e^i , \\
\omega_{mn} &= \frac{1}{4} Z^{-1/4} (\partial_n \log Z e^m - \partial_m \log Z e^n) , \\
\omega_{+m} &= \frac{1}{4} Z^{-1/4} \partial_m \log Z e^- + \frac{1}{2} Z^{-1/4} \partial_m \log N e^+ , \quad \omega_{+i} = \frac{1}{2} Z^{1/4} \partial_i \log N e^+ , \quad (18)
\end{aligned}$$

We only need to consider gravitino variations given the nontrivial fields in these backgrounds. Then  $\delta\psi_M = \frac{1}{\kappa} D_M \epsilon + \frac{i}{480} \gamma^{M_1 \dots M_5} F_{M_1 \dots M_5} \gamma_M \epsilon = 0$ , shows that the variations  $\delta\psi_-, \delta\psi_i, \delta\psi_m$  are the same as for the case  $N = 0$ . Evaluating  $\delta\psi_+$  gives new terms containing  $(\omega_{+m+} \Gamma^{+m} + \omega_{+i+} \Gamma^{+i}) \epsilon$  in addition to the terms for  $N = 0$ . All these conditions can thus be satisfied if the spinor satisfies  $\Gamma^4 \epsilon = \epsilon$  (as for the usual D3-brane solution), and  $\Gamma^+ \epsilon = 0$ . The latter is the familiar spinor condition for null solutions. What we thus see is that the inhomogenous  $AdS$  plane waves preserve just as much supersymmetry as the homogenous one.

### 3 On $x^+$ -dimensional reduction

The generic 10-dim solution (2) can be dimensionally reduced on  $X^5$  to give a 5-dim system: with  $g_{++}(r, x_i)$  having no  $x^+$ -dependence, this is a solution to 5-dim gravity with negative cosmological constant (and no other matter sources). Now we consider regarding  $x^+$  as a compact direction and dimensionally reducing  $\int d^5 x \sqrt{-g^{(5)}} (R^{(5)} - 2\Lambda)$  on it as

$$\int dx^+ d^4 x \sqrt{-g^{(4)}} (R^{(4)} - \# \Lambda e^{-\phi} - \# (\partial\phi)^2 - \# e^{3\phi} F_{\mu\nu}^2),$$

where the 4-dim metric undergoes a Weyl transformation as  $g_{\mu\nu}^{[4]} = e^\phi g_{\mu\nu}^{[5]}$  (and the numerical constants  $\#$  can be fixed). This gives the resulting 4-dim Einstein metric (relabelling  $x^- \equiv t$ ) as

$$ds_4^2 = \sqrt{g_{++}} \left( -\frac{dt^2}{r^4 g_{++}} + \frac{dx_i^2 + dr^2}{r^2} \right), \quad e^\phi = \sqrt{g_{++}}, \quad A_t = -\frac{dt}{r^2 g_{++}}, \quad (19)$$

with  $g_{++}(r, x_i)$ , and the overall conformal factor  $\sqrt{g_{++}}$  in the metric arises from the KK-scalar  $e^\phi$ . For the homogenous case with  $g_{++} = Qr^2$ , *i.e.* the  $AdS$  plane wave, discussed in [12], this gives the hyperscaling violating metric lying in the family  $\theta = d - 1$ . The KK scalar grows in the interior as  $\phi \sim \log r$  in this case. All normalizable  $g_{++}$  solutions have  $g_{++} \rightarrow_{r \rightarrow 0} r^2 f_4(x_i)$  so their radial dependence near the boundary is similar to the homogenous  $AdS$  plane wave case: near  $r \rightarrow 0$ , the lower dimensional description (19) approaches

$$ds_4^2 = -\frac{dt^2}{r^5 \sqrt{f_4(x_i)}} + \frac{\sqrt{f_4(x_i)}}{r} (dx_i^2 + dr^2), \quad e^\phi = r \sqrt{f_4(x_i)}, \quad A_t = -\frac{dt}{r^4 f_4(x_i)}. \quad (20)$$

These static solutions from the lower dimensional point of view are of course of a specific kind (*e.g.* gauge field being solely electric), with some function  $g_{++}(r, x_i)$  introducing inhomogeneities. Presumably more general inhomogeneous backgrounds can be found in the lower dimensional theory: their uplift might not fit into this *AdS* null deformation pattern upstairs of course.

The more general inhomogeneous cases discussed here give rise to inhomogeneous solutions after dimensional reduction too. However in these cases, several issues arise generically, the most basic one being that  $g_{++} \rightarrow 0$  somewhere in the interior implying that the  $x^+$ -circle shrinks. A little investigation suggests that generically, the surface  $g_{++} = 0$  (upstairs) does not give rise to a smooth cigar-like geometry in the  $(r, x^+)$ -directions: generically  $g_{++} = 0$  does not also coincide with  $\partial_r g_{++} = 0$ . For concreteness, assuming a Fefferman-Graham-like expansion  $g_{++} \sim (r - r_0)^2 f_2 + \dots$ , for  $g_{++}$  exists near  $r = r_0(x_i)$  where  $g_{++}(r_0) = 0$ , the equation of motion (4) is not satisfied near  $r = r_0$  unless  $f_2 = 0$ . These seem worse than conical singularities in the interior, with  $g_{++} \sim c(r - r_0) + \dots$  in the vicinity of the horizon  $g_{++} = 0$ . This is not in contradiction with the fact that all curvature invariants vanish everywhere (upstairs), which is a consequence of the lightlike nature of this entire class of solutions. Thus these configurations generically do not seem like analogs of the *AdS*-soliton where the geometry caps off smoothly at some  $r = r_0$ . However this is worth exploring further.

For  $x^+$ -compact, we have a shrinking circle, in which vicinity  $g_{++} \sim c(r - r_0) + \dots$ : this means that in the vicinity of this location, string modes winding around the  $x^+$ -circle become light so that the gravity solution cannot be trusted. From the dual field theory perspective, various operators dual to these string states acquire low anomalous dimensions so do not decouple from the effective low energy dynamics. It would seem that these new light states would significantly alter the system. Assuming that the effects of these light states are approximately localized and do not destabilize the entire spacetime, one might imagine that the metric in (19) is approximately valid away from the locations where the circle pinches off. In general, this gives inhomogeneous domain-like structures in the  $x_i$ -plane. For concreteness, consider the 5-dim spacetime in (11) (ignoring the  $X^5$ ), which has horizons as  $g_{++} \rightarrow 0$  in the spatial domains where  $\sin kx < 0$ . Then near the horizon, we have  $g_{++} \sim c(r - r_0) + \dots$ , where  $r_0(x_i)$  in general varies as a function of the spatial location  $x_i$ : from (19), we then have

$$ds_4^2 \sim \sqrt{r - r_0} \left( -\frac{dt^2}{r^4(r - r_0)} + \frac{dx_i^2 + dr^2}{r^2} \right), \quad e^\phi \sim \sqrt{r - r_0}, \quad A_t \sim -\frac{dt}{r^2(r - r_0)}. \quad (21)$$

Similarly, in the spatial domains where  $g_{++} > 0$  everywhere, we have  $g_{++} \rightarrow_{r \rightarrow \infty} \frac{e^{kr}}{\sqrt{r}} \sin kx$ : then the asymptotic lower dimensional metric and KK-scalar from (19) are

$$ds_4^2 \rightarrow_{r \rightarrow \infty} -\frac{dt^2}{r^{9/2} e^{kr/2} \sqrt{\sin kx}} + \frac{e^{kr/2}}{r^{5/2}} \sqrt{\sin kx} (dx_i^2 + dr^2) , \quad e^\phi \sim \frac{e^{kr/2}}{r^{1/4}} \sqrt{\sin kx} . \quad (22)$$

There is in general a lattice-like structure in the  $x_i$ -plane for  $g_{++}(r, x_i)$  having periodic structure in the  $x_i$ : the locations where  $g_{++} = 0$  define unit cell boundaries in a sense. Near these boundaries, we have  $g_{tt} \rightarrow \frac{1}{\sqrt{g_{++}}}$  while  $g_{ii}, g_{rr} \rightarrow \sqrt{g_{++}} \rightarrow 0$ , implying that the spatial directions shrink, while the time direction grows, *i.e.* lightcones open up.

### 3.1 Asymptotically Lifshitz solutions with inhomogeneities

Consider now  $g_{++}(x^+, x_i, r)$  to be a function of  $x^+$  also: then the equation for  $g_{++}$  continues to be (4), which most notably has no  $x^+$ -derivatives. This means one can separate variables here, implying that *AdS* deformations with metric of the form (2), and

$$g_{++} = F(x^+) + g_{++}(x_i, r) , \quad (23)$$

are all solutions, with the non-normalizable deformation  $F(x^+)$  being sourced through  $R_{++}$  by various lightlike sources contributing to  $T_{++}^{bulk}$ . The leading term  $F(x^+)$  (which could be an  $x^+$ -independent constant as well) by itself gives rise, after  $x^+$ -dimensional reduction, to  $z = 2$  4-dim Lifshitz spacetimes as discussed in the string constructions [5, 6] holographically dual to the corresponding deformations (and DLCQ) of SYM theories. Then, if we restrict to normalizable solutions in the second term  $g_{++}(x_i, r)$  as in the previous sections, we have  $g_{++} \sim_{r \rightarrow 0} F(x^+) + r^2 f_4(x_i)$ : in this case, these solutions after  $x^+$ -dimensional reduction, are all asymptotically Lifshitz. Thus they are best thought of as states in the Lifshitz-like (non-normalizable) null-deformations (and DLCQ) of the SYM theory.

We now first describe some aspects of these Lifshitz string constructions involving *AdS* null deformations, focussing therefore only on the leading non-normalizable term  $F(x^+)$  in  $g_{++}$ : this gives some *AdS/CFT* perspective on the Lifshitz vacuum in these constructions. First we note that this term  $F(x^+)$  does not contribute at all to the holographic stress tensor in these backgrounds, giving  $T_{++} = 0$  (including counterterms for the sources): *i.e.* the holographic stress tensor vanishes despite the fact that there are nontrivial lightlike matter fields (*e.g.* a null dilaton) sourcing the null deformation. To elaborate, consider first the case [5] where  $F(x^+)$  is sourced by a lightlike scalar (*e.g.* the dilaton): then the stress tensor, using [36], is

$$T_{\mu\nu} \sim \frac{1}{G_5} \left( K_{\mu\nu} - K h_{\mu\nu} - 3 h_{\mu\nu} + \frac{1}{2} G_{\mu\nu} - \frac{1}{4} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{8} h_{\mu\nu} (\partial \Phi)^2 \right) , \quad (24)$$

where  $G_{\mu\nu}, \partial$  here are defined w.r.t. the boundary metric  $h_{\mu\nu}$ , and the scalar terms arise from counterterms involving the scalar (and  $K_{\mu\nu}$  is the extrinsic curvature as before, see footnote 1)<sup>2</sup>. The stress tensor can then be seen to vanish if, in (23), the leading term  $F(x^+)$  alone is nonvanishing.

Alternatively, using the coordinate transformation  $r = we^{-f/2}$ ,  $x^- = y^- - \frac{r^2 f'}{4}$ , the metric (2) with *e.g.*  $g_{++} = \frac{1}{4}(\partial_+ \Phi)^2$  can be recast as

$$ds^2 = \frac{1}{w^2} [e^{f(x^+)} (-2dx^+ dy^- + dx_i^2) + dw^2] + d\Omega_5^2, \quad \Phi = \Phi(x^+), \quad (25)$$

with the constraint  $\frac{1}{2}(\partial_+ f)^2 - \partial_+^2 f = \frac{1}{2}(\partial_+ \Phi)^2$ . In this form, comparing with the Fefferman-Graham expansion (5), the metric can be seen to have  $g_{\mu\nu}^0 = e^{f(x^+)} \eta_{\mu\nu}$  with all subleading coefficients identically vanishing: this solution is part of the general family of solutions

$$ds^2 = \frac{1}{r^2} (\tilde{g}_{\mu\nu}(x^\mu) dx^\mu dx^\nu + dr^2) + d\Omega_5^2, \quad \Phi = \Phi(x^\mu), \quad (26)$$

(after relabelling  $w$  as  $r$ ) with the  $X^5$  and 5-form suppressed, and the conditions  $\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu \Phi \partial_\nu \Phi$ ,  $\square \Phi = 0$  [41, 36] arising from the IIB supergravity equations of motion. These conditions can also be obtained using holographic renormalization methods [32, 33] and requiring that all subleading coefficients vanish from the Fefferman-Graham expansion for both metric (5) and scalar  $\Phi = r^{(d-\Delta)/2}(\Phi^0 + r^2 \Phi^2 + \dots)$ , by solving  $R_{MN} = -4g_{MN} + \frac{1}{2}\partial_M \Phi \partial_N \Phi$  iteratively: this gives

$$g_{\mu\nu}^2 \sim R_{\mu\nu}^0 - \frac{1}{2}\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2(d-1)} \left( R - \frac{1}{2}(\partial\Phi)^2 \right) g_{\mu\nu}^0 : \quad g_{\mu\nu}^2 = 0 \Rightarrow R_{\mu\nu}^0 = \frac{1}{2}\partial_\mu \Phi \partial_\nu \Phi, \quad (27)$$

(for a massless scalar  $\Delta = d$ ) also implying the higher order coefficients vanish if  $g^{(4)} = 0$ . Likewise, with  $\square^0$  being the Laplacian w.r.t.  $g_{\mu\nu}^0$ , we also obtain  $\Phi^{(2)} \sim \square^0 \Phi^0$ : thus  $\Phi^{(2)} = 0$  implies  $\square^0 \Phi^0 = 0$ . In these coordinates, the boundary metric is  $h_{\mu\nu} = \frac{1}{r^2} \tilde{g}_{\mu\nu}$ , we see that the extrinsic curvature is  $K_{\mu\nu} = -h_{\mu\nu}$ , so that the first three terms in  $T_{\mu\nu}$  given in (24) cancel identically while the the last three terms also cancel using the constraint relation  $\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu \Phi^{(0)} \partial_\nu \Phi^{(0)}$ . Thus the stress tensor vanishes identically for these leading non-normalizable deformations  $g_{++} = F$ .

The solutions (25), (26), thus appear constrained from this point of view, with only the first coefficient  $g^{(0)}$ ,  $\Phi^{(0)}$  nonzero for all  $r$ , the subleading pieces of the metric and scalar vanishing. These conditions on the  $g_{\mu\nu}^n$ ,  $\Phi^n$ ,  $n > 0$ , are non-generic and appear to be nontrivial constraints fine-tuning the CFT state after turning on the sources  $g^{(0)}$ ,  $\Phi^{(0)}$ .

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<sup>2</sup>See *e.g.* [37, 38, 39, 40] for recent discussions of holographic renormalization in Lifshitz backgrounds; see also [35].

These arguments have also been discussed in [42] in the cosmological singularities context of [41, 36].

This discussion on the Fefferman-Graham expansion is mainly based on the foliation where we have  $g^{(0)} \neq 0$  alone: however the relation between  $g^{(2)}, \Phi^{(0)}$  can be seen to be true even otherwise. Although we have presented these arguments for the case where  $F(x^+)$  is sourced by a lightlike dilaton, it appears likely that similar arguments can be made for other matter sources discussed in [6], using appropriate matter counterterms to construct the holographic stress tensor free of ultraviolet divergences. From this point of view, the Lifshitz vacuum has been obtained by deforming using lightlike sources and also setting the stress tensor to vanish (by requiring vanishing of *e.g.*  $g_{\mu\nu}^{(4)}$ ), thereby tweaking the state too.

It is worth making some general comments here. As discussed in [36], the coordinate transformations above, recasting the metric to have a conformally flat boundary, are Penrose-Brown-Henneaux (PBH) transformations [43, 44, 45, 46, 47]: these are bulk diffeomorphisms acting as a Weyl transformation on the boundary. In general, a PBH transformation changes the bulk foliation and thus the near-boundary Fefferman-Graham expansion. In particular, a spacetime where the boundary metric  $g_{\mu\nu}^{(0)}$  is conformally flat, as in (25), (26), will generically have nonzero subleading coefficients  $g_{\mu\nu}^{(n)}$ ,  $n > 0$ , after a PBH transformation: in general, this will lead to a nonvanishing stress tensor as well, as studied in some of the time-dependent situations in [36]. From this point of view, the stress tensor continuing to vanish for the null deformations here is in some sense accidental, stemming from the fact that the PBH transformation here results in a finite series, the metric in the PBH coordinates (2) truncating at order  $g_{++}^{(2)}$ . Therefore the above arguments on constraining the state appear foliation-dependent. Perhaps it is fair to say that the existence of a coordinate choice, or foliation, where the Fefferman-Graham expansion appears constrained (*e.g.* with the corresponding holographic stress tensor vanishing) is not generic. It would be worth understanding these issues better.

This appears to have some bearing on the Lifshitz singularity due to diverging tidal forces as  $r \rightarrow \infty$  [48]. In the conformal coordinates (26), we see that the *AdS* deformation with  $g_{\mu\nu}^{(0)} = \tilde{g}_{\mu\nu}$  alone could potentially lead to singularities on the Poincare horizon  $r \rightarrow \infty$ , with the curvature invariants  $R \sim r^2 \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + O(r^0)$ ,  $R_{ABCD} R^{ABCD} \sim r^4 \tilde{R}_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta} + O(r^0)$  etc diverging (this can be seen by expanding out the curvature components and using the equation  $\tilde{R}_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi$ : see also [42] in the cosmological singularities context [41, 36]). Equivalently, a metric that is regular everywhere, with an expansion of this form about  $r = 0$ , must have the coefficients  $g_{\mu\nu}^{(n)}$  generically nonzero if a singularity is to be avoided at large  $r$ . For the null metric (25), these curvature invariants vanish, since the lightlike solutions admit no nonzero contraction. However this is a common feature of lightlike solutions, as we may

recall from plane-wave spacetimes. The singularities then arise from diverging tidal forces. In this form, we see that the divergence is due to the fact that the metric is constrained with all subleading coefficients vanishing. However it is worth noting that the state in the *AdS/CFT* perspective appearing constrained is not automatically equivalent to the string background being singular. Since the Lifshitz vacuum has been constructed by turning on nontrivial sources, it would appear that strings would experience nontrivial scattering due to the “stuff” making up the sources, rendering the metric contribution alone likely incomplete: this has been discussed in [49] in the context of the Lifshitz string constructions in [50]. In this light, the Lifshitz backgrounds above are also likely to be good string backgrounds, even if they appear constrained from an *AdS/CFT* point of view at the level of supergravity (see also [51] for other related discussions on Lifshitz singularities). For instance, in the case [5] where  $F(x^+) \sim (\partial_+ \Phi)^2$  is sourced by a lightlike dilaton  $\Phi(x^+)$  arising from the NS-NS sector alone, the source would appear to be a condensate of background strings making up the dilaton profile, while the more general sources in [6] include R-R backgrounds. It would be interesting to understand string propagation in these backgrounds better. Relatedly it would also be interesting to obtain a deeper understanding of the rules of *AdS/CFT* and its deformations/states for such apparently constrained backgrounds.

An interesting question in this regard pertains to the existence of inhomogenous phases in asymptotically Lifshitz backgrounds, with broken spatial translation invariance. In the context of *AdS* null deformations, this can be simulated by  $g_{++}$  modes of the form (23), which are in the full theory (not just in linearized gravity). Once we turn on the subleading normalizable modes in (23), the stress tensor acquires a nonzero expectation value  $T_{++} \sim \frac{f_4(x_i)}{G_5}$ , where  $g_{++}(r, x_i) = F + r^2 f_4 + \dots$ . These solutions have the same feature as before, *i.e.*  $g_{++} = 0$  somewhere in the interior. After the  $x^+$ -dimensional reduction, this means that there are light string winding states that arise in the vicinity of these loci, which therefore suggest new stringy physics that modifies the gravity solution. It would be interesting to explore these further.

Besides null deformations (23), it is of interest to look for different kinds of inhomogenous backgrounds, in particular say striped phases. In this regard, note that technically, a 4-dim Lifshitz spacetime is a solution to 4-dim gravity with negative cosmological constant coupled to a massive vector: this is similar to the field content for *e.g.* charged 4-dim *AdS* black brane solutions, so one might imagine technical similarities between the two systems in the analysis of linearized gravity perturbations, discussed for the latter in [52]. In the Lifshitz context, the massive gauge field and metric lift in the 5-dim metric-dilaton system to metric modes, and we can then look for perturbations restricting to simply normalizable metric modes “upstairs” (the scalar takes on simply its non-normalizable background profile). Then for instance,



it can be seen that the modes  $\{g_{+y}(x^+, x, r), g_{-y}(x, r)\}$  give rise to a closed subsystem of equations at the linearised level. These map in the lower dimensional description to  $\{A_y, g_{ty}\}$ . Consider an explicit ansatz for striped phases of the form (with  $F(x^+) = \frac{1}{4}(\partial_+ \Phi)^2$  as in the leading solution),  $g_{+y} = \frac{1}{r^2} F h_{+y}(r) \sin kx$ ,  $g_{-y} = \frac{1}{r^2} h_{ty}(r) \sin kx$ , that simulate translation-invariance breaking states: this mimics possible regular clumps along the  $x$ -direction while translation invariance continues to exist along the  $y$ -direction. Then the equations of motion at linear order simplify to  $h''_{ty} - \frac{3}{r} h'_{ty} - k^2 h_{ty} = 0$ ,  $h''_{+y} - \frac{3}{r} h'_{+y} - k^2 h_{+y} = -2r h'_{ty}$ , from the  $R_{-y}, R_{+y}$ -equations respectively, which are now nontrivial. These equations at linear order are similar to (8) for the  $g_{++}$ -backgrounds we have discussed. The solution  $h_{ty} \sim r^2 I_2(kr)$  that is normalizable at the boundary  $r = 0$  (with asymptotics  $h_{ty} \sim r^4$ ,  $r \rightarrow 0$ ) grows in the interior as  $h_{ty} \sim e^{kr}$  ( $r \rightarrow \infty$ ). This  $h_{ty}$ -solution also implies a solution for  $h_{+y}$ . These solutions exist for any  $k > 0$ , and suggest striped phases in the IR (it is unclear at this level if there is any critical  $k$ ). It can be checked that these solutions obtained at the linearised level are not solutions in the full system, which appears more complicated: the fact that these modes grow in the interior might suggest the breakdown of linear perturbation theory here, unlike the  $g_{++}$ -mode. Likewise, the modes  $\{g_{+-}(x^+, x, r), g_{++}(x^+, x, r), g_{yy}(x, r)\}$  give rise to a closed subsystem of equations at the linearised level: the ansatz  $g_{+-} = -\frac{1}{r^2}(1 + h_{+-}(r) \sin kx)$ ,  $g_{++} = F(1 + h_{++}(r) \sin kx)$ ,  $g_{yy} = \frac{1}{r^2}(1 + h_{yy}(r) \sin kx)$ , give at linear order the equations  $h_{yy} = -2h_{+-}$ ,  $h''_{+-} - \frac{3}{r} h'_{+-} - k^2 h_{+-} = 0$ ,  $r^2 h''_{++} + r h'_{++} - (4 + k^2 r^2) h_{++} = 4r h'_{+-}$ . As above,  $g_{+-}$  satisfies the same equation as for a massless scalar. These sorts of modes suggest different kinds of striped IR phases, albeit at linear order only, in asymptotically Lifshitz spacetimes, which are not null deformations. It would be interesting to explore these further.

## 4 Holographic entanglement entropy

We want to understand holographic entanglement entropy of subsystem  $A$  in the boundary field theory dual to an  $AdS$  null deformation of the form (2), using the prescription of [10] of finding the area of a bulk minimal surface bounding the subsystem  $A$ . We will primarily discuss the case where subsystem  $A$  has the shape of a strip in the  $x, y$ -plane. Since these systems are naturally described in terms of slicings at constant lightcone time  $x^-$  (rather than a timelike coordinate), it is natural to consider a minimal surface at a constant- $x^-$  slice. The spatial metric on such a slice then is

$$ds^2 = \frac{R^2}{r^2} (dx_i^2 + dr^2) + g_{++}(r, x_i) (dx^+)^2, \quad (28)$$

where we have reinstated the  $AdS$  radius  $R$ . Subsystems  $A$  that straddle some part of the  $x_i$ -plane but extend along the  $x^+$ -direction completely are natural from the point of view of the lower dimensional theory obtained by dimensional reduction along the  $x^+$ -direction, as is implicit in [12]: this then gives in the bulk a minimal surface wrapping the  $x^+$ -direction and bounding the subsystem  $A$  in the  $x_i$ -plane. If  $g_{++} = 0$ , this surface degenerates and becomes null, as is clear from the spatial metric above. Consider a strip region in the  $x$ -direction given by  $-l \leq x \leq l$ , extending along the  $y$ -direction: then we expect that the minimal surface is parametrized by  $r = r(x)$ , and its area gives the entanglement entropy

$$S_E = \frac{1}{G_5} \int_0^L \frac{R dy}{r} \int_0^{L_+} \sqrt{g_{++}} dx^+ \int \frac{R \sqrt{dx^2 + dr^2}}{r} = \frac{LL_+ R^2}{G_5} \int_\epsilon dr \frac{\sqrt{g_{++}}}{r^2} \sqrt{1 + \left(\frac{dx}{dr}\right)^2}, \quad (29)$$

where  $\epsilon$  is the near-boundary cutoff (*i.e.* the UV cutoff in the field theory). The minimal surface has a turning point  $r_t$  where  $\frac{dr}{dx}|_{r_t} = 0$ . Let us now consider homogenous backgrounds, to begin with. Then  $g_{++}$  has no  $x_i$ -dependence, giving the usual conserved conjugate momentum to  $x(r)$ , which enables us to solve for the minimal surface, its area and the entanglement entropy. For instance, for  $z = 2$  Lifshitz backgrounds,  $g_{++}$  has no  $r, x_i$ -dependence: taking it to be constant for simplicity, we have  $S_E \sim \frac{LL_+}{G_5} \int_\epsilon dr \frac{1}{r^2} \sqrt{1 + (x')^2}$ . After  $x^+$ -compactification, with  $\frac{G_5}{L} = G_4$ , this is of the same form as the lower dimensional expression for a 4-dim Lifshitz spacetime (as well as  $AdS_4$  with constant time slices). The conserved momentum  $\frac{1}{r^2} \frac{x'}{\sqrt{1+(x')^2}} = p$  gives the turning point  $r_t = \frac{1}{\sqrt{p}}$ , the equation for the surface  $\frac{dx}{dr} = \frac{1}{\sqrt{1-p^2 r^4}} - 1$ , thereby  $l = r_t$ , and

$$S_E \sim \frac{LL_+ R^2}{G_5} \int_\epsilon^{r_t} dr \frac{1}{r^2 \sqrt{1 - (r^4/r_t^4)}} \sim \frac{LL_+ R^2}{G_5 l} \int_{\epsilon/l}^1 du \frac{1}{u^2 \sqrt{1 - u^4}} \sim \frac{L_+ R^2}{G_5} \left( \frac{L}{\epsilon} - \# \frac{L}{l} \right), \quad (30)$$

where the coefficient in the second term arises from  $\int_0^1 du \frac{1}{u^2} \left( \frac{1}{\sqrt{1-u^4}} - 1 \right)$  which has a finite value. The first term is a reflection of the area law. This calculation is in fact technically the same as that for  $AdS_4$  with constant time slices (see *e.g.* [10]). It is worth emphasizing that this nonvanishing entanglement entropy for the Lifshitz state is due to a nonvanishing  $g_{++}$ : using constant- $x^-$  slices for the  $AdS_5$  vacuum gives a vanishing result, since  $g_{++} = 0$  (while using slices of a timelike time coordinate gives the usual expressions respecting the area law).

For the homogenous  $AdS$  plane wave considered in [12], we have  $g_{++} = R^2 Q r^2$ , giving

$$S_E = \frac{LL_+ R^3 \sqrt{Q}}{G_5} \int_\epsilon^{r_t} dr \frac{\sqrt{1 + (x')^2}}{r}; \quad \frac{1}{r} \frac{x'}{\sqrt{1 + (x')^2}} = p, \quad l = r_t = \frac{1}{p}. \quad (31)$$

The minimal surface then is  $x = l \sqrt{1 - \left(\frac{r}{r_t}\right)^2}$ , and we have  $S_E = \frac{LL_+ R^3 \sqrt{Q}}{G_5} \int_{\epsilon/l}^1 \frac{du}{u \sqrt{1 - u^2}}$ ,

giving

$$S_E = \frac{2L_+ R^3 \sqrt{Q}}{G_5} L \log \frac{l}{\epsilon} . \quad (32)$$

Using  $G_4 = \frac{G_5}{L_+}$ , this gives the logarithmic violation of the area law, as expected from the lower dimensional theory. We have effectively taken  $l \gg \epsilon$ , so that the strip width  $l$  is macroscopic relative to the UV cutoff  $\epsilon$  in the field theory. When the strip size shrinks to roughly the cutoff, we have a cross-over to the UV behaviour in the field theory: in this case, we expect the entanglement entropy for  $AdS_5$  in lightcone time slicing which vanishes, as vindicated by (32) for  $l \sim \epsilon$ .

Note that (30) and (32) both agree with the corresponding expressions obtained from the lower dimensional descriptions, in the the 4-dim  $z = 2$  Lifshitz spacetime and the hyperscaling violating metric with  $\theta = 1, d = 2$  respectively. The latter was of course expected from the dimensional reduction in [12]. It is interesting that this calculation (32) above arises from just the 5-dim part of the spacetime, so in particular it also applies to  $AdS_5 \times X^5$  homogenous plane waves dual to various  $\mathcal{N}=1$  super Yang-Mills theories.

Consider now an asymptotically Lifshitz solution superposed with the homogenous plane wave state  $g_{++} = F + R^2 Q r^2$ . We then expect the entanglement entropy to exhibit a cross-over from Lifshitz ground state behaviour to the logarithmic violation above.

Now let us consider inhomogenous backgrounds: for simplicity, we will focus here on backgrounds which preserve  $y$ -translation symmetry, *i.e.* where  $g_{++}$  has no  $y$ -dependence. Then consider a strip in the  $x$ -direction given by  $-l \leq x \leq l$ , extending along the  $y$ -direction: we expect that the minimal surface in this case is still parametrized by  $r = r(x)$  on symmetry grounds. The entanglement entropy is then given by (29) as before. If the strip region is symmetric w.r.t. the bulk geometry, we expect the minimal surface to have a turning point  $r_t$  where  $\frac{dr}{dx}|_{r_t} = 0$ . In general, regarding (29) as an action for  $x(r)$ , the Euler-Lagrange equation of motion for extremization gives ( $x' \equiv \frac{dx}{dr}$ )

$$\frac{d}{dr} \left( \frac{\sqrt{g_{++}}}{r^2} \frac{x'}{\sqrt{1 + (x')^2}} \right) = \frac{\sqrt{1 + (x')^2}}{2r^2 \sqrt{g_{++}}} \frac{\partial g_{++}}{\partial x} . \quad (33)$$

Now consider a spacetime of the form (2) where  $g_{++}$  is a “small” inhomogenous departure from the homogenous  $AdS$  plane wave, in the sense of (13), *i.e.* with  $T_{++} \sim Q + \epsilon f(x)$ . Then we expect that at least for a strip of small width (and centered about the lump at  $x = 0$ ), the minimal surface will not dip into the interior too much and so will not be sensitive to the horizon: this will again give the entanglement entropy to be (32) approximately. For larger strip widths, the minimal surface will dip further into the interior and will exhibit significant departures. Perhaps analysing entanglement entropy further will give more insight into such backgrounds.

## 5 Conclusions

We have described *AdS* analogs of plane waves with possible inhomogeneities. These are likely to be  $\alpha'$ -exact string backgrounds. They correspond in the dual gauge theory to turning on lightcone momentum density  $T_{++}$  varying spatially. We have seen that generically the inhomogeneous *AdS* plane waves exhibit analogs of horizons where a timelike Killing vector becomes null. From the point of view of  $x^+$ -dimensional reduction, the circle pinches off at these horizon loci, string winding modes becoming light here, indicating new stringy physics.

In general these are normalizable backgrounds (rather than fluctuations) and grow in the interior, as we have discussed. One might wonder if these *AdS* plane waves with inhomogeneities are unstable towards “settling” down to the homogeneous one with uniform  $T_{++}$  studied in [12]. However all these backgrounds have finite curvature invariants, and preserve as much supersymmetry as the homogeneous case. It would be interesting to understand this better, including finite temperature versions.

We have also described asymptotically Lifshitz backgrounds of this sort with possible inhomogeneities, with some discussion of Lifshitz singularities from the perspective of the *AdS/CFT* construction involving null deformations. The Lifshitz vacuum appears to be a non-generic state in these constructions. It would be interesting to obtain a deeper understanding of string propagation in such backgrounds, as well as the rules of *AdS/CFT* and its deformations/states for such backgrounds.

Finally it is interesting to explore holographic entanglement entropy in these backgrounds, especially from the point of view of the dual field theory. We recall that the homogeneous background (for *AdS*<sub>5</sub>) with uniform lightcone momentum density [12] interestingly exhibits logarithmic violation of the area law of entanglement entropy. We hope to explore this further [53].

**Acknowledgments:** It is a pleasure to thank K. Balasubramanian, S. Das, G. Horowitz, N. Iqbal, S. Kachru, S. Minwalla, M. Rangamani, E. Silverstein, N. Suryanarayana, T. Takayanagi and S. Trivedi for helpful conversations on various aspects of this work. I also thank the hospitality of the KITP Santa Barbara, Stanford Institute for Theoretical Physics, Physics Dept, U. Kentucky, USA, the Organizers of the String Theory Discussion Meeting, June '12, at the International Center for Theoretical Sciences (ICTS), Bangalore, and the String Theory groups at TIFR, Mumbai, at various stages while this work was being carried out. This work is partially supported by a Ramanujan Fellowship, DST, Govt. of India.

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